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Planar field theories with space-dependent noncommutativity

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Abstract

We study planar noncommutative theories such that the spatial coordinates $\hat{x}_1, \hat{x}_2$ verify a commutation relation of the form: $[\hat{x}_1, \hat{x}_2] = \mathrm{i}\theta(\hat{x}_1, \hat{x}_2)$. Starting from the operatorial representation for dynamical variables in the algebra generated by $\hat{x}_1$ and $\hat{x}_2$, we introduce a noncommutative product of functions corresponding to a specific operator-ordering prescription. We define derivatives and traces, and use them to construct scalar-field actions. The resulting expressions allow one to consider situations where an expansion in powers of θ and its derivatives is not necessarily valid. In particular, we study in detail the case when θ vanishes along a linear region. We show that, in that case, a scalar field action generates a boundary term, localized around the line where θ vanishes.

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1. Introduction

Noncommutative quantum field theories[1] have recently attracted renewed attention, not only because of their relevance to string theory [1, 3], but also in the condensed matter physics context, since they have been proposed as effective descriptions of the Laughlin states in the quantum Hall effect [4–6]. Noncommutativity has also been introduced to describe the skyrmionic excitations of the quantum Hall ferromagnet at $\nu = 1$ [7, 8].

A planar system of charged particles in the presence of an external magnetic field has a very rich structure, in part because of the peculiarities of the Landau level spectrum for a single particle [9]. A noncommutative description is usually invoked as a way to describe a restriction to the lowest Landau level, a step which is justified by the existence of a large gap between the lowest and higher Landau Levels [10, 11]. This restriction cannot be introduced as a smooth limit of the full (all level) system, since there is a change in the number of

[1] See, for example, [1, 2] for pedagogical reviews.

physical degrees of freedom, an effect that has been known since the early studies on Chern–Simons quantum mechanics [12], and which is entirely analogous to the reduction from the Maxwell–Chern–Simons action into the pure Chern–Simons theory [12, 13].

In this paper, we address the problem of describing planar noncommutative theories where θ, the noncommutativity parameter, is a space-dependent object. If the dependence of θ is sufficiently smooth, this phenomenon can be studied within the deformation quantization approach [14], since it naturally allows for an expansion in powers of θ and its derivatives. We are, however, interested in the cases where θ is not necessarily smooth, namely, when θ may have an appreciable variation over length scales of the order of $\sqrt{\theta}$. For example, one may think of situations where θ has first-order zeros in a certain region of the plane.

It is important to have the tools to describe that sort of situation, since it may naturally occur in the condensed matter physics context. For example, when the relation between the magnetic field and the effective mass is space dependent; or one could want to study interfaces that divide regions with different noncommutativity parameters. If that interface is rather narrow, an expansion in powers of θ and its derivatives will certainly be unreliable.

A way to deal with the case of a θ that depends on only one of the variables has been presented in [15]. Our approach is instead based on the use of a particular mapping between the operatorial representation of the theory, and its functional version. We construct the noncommutative theory in a way that is in principle valid for a more general θ, although explicit results are presented for the case of a θ that depends on one variable.

The analysis of theories with space-dependent noncommutativity has been full of technical difficulties, both at the mathematical and physical levels. Much effort has been devoted in recent years to understanding their fundamental properties. In [16] Kontsevich's construction is interpreted in terms of a path integral over a sigma model. On the other hand, the relation between the noncommutativity function and curved branes in curved backgrounds has been studied in [17]. Besides, in [18], it is shown how to construct $U(1)$ gauge-invariant actions when the noncommutativity is space dependent. See also [19] for a formulation of gauge theories on spaces where the commutator between space coordinates is linear or quadratic in those coordinates.

The structure of this paper is as follows: in section 2 we set up the general framework, defining the elements that are required to construct the noncommutative field theory in the operatorial version of the algebra. In section 3, we deal with the representation of the theory in its functional form, namely, using functions with a $\star$-product. These general results are applied, in section 4, to the case in which $\theta(\hat{x})$ is an invertible operator, and depends on $\hat{x}_1$ and $\hat{x}_2$ only through a linear combination of them, i.e., $\theta(\hat{x}_1, \hat{x}_2) = \theta(c_1\hat{x}_1 + c_2\hat{x}_2)$. In section 5, for a $\theta(\hat{x})$ with an analogous dependence, we allow for a null eigenvalue, and discuss the physical consequences of that property. Finally, in section 6, we present our conclusions.

2. Operatorial description

We shall consider quantum field theories defined on a two-dimensional noncommutative region generated by two elements, $\hat{x}_1$ and $\hat{x}_2$, which satisfy a local commutation relation

$$[\hat{x}_j, \hat{x}_k] = \mathrm{i}\epsilon_{jk}\theta(\hat{x}), \tag{1}$$

where $j, k = 1, 2$ and $\hat{x}_1, \hat{x}_2$ denote Hermitian operators on a Hilbert space $\mathcal{H}$. $\theta(\hat{x})$, also a Hermitian operator, is a local function of $\hat{x}_1$ and $\hat{x}_2$. The form of θ will be further restricted later on, when considering some particular examples. That will allow us to derive more explicit results, at least under simplifying assumptions.

We are interested in defining field theory actions for fields that belong to the space $\mathcal{A}$, the algebra generated by $(\hat{x}_1, \hat{x}_2)$. To accomplish that goal, one has to introduce two independent derivations (corresponding to the two coordinates $\hat{x}_1$ and$\hat{x}_2$) plus an integration on $\mathcal{A}$. Since the fields, their products and linear combinations, are all elements of $\mathcal{A}$, one can, as usual, define the integral as the trace of the corresponding product of fields. Indeed,

$$\int A(\hat{x}_i) \equiv \mathrm{Tr}[A(\hat{x}_i)], \tag{2}$$

for any $A \in \mathcal{A}$, has the property of being linear and invariant under cyclic permutation of the factors in a product.

Regarding the derivatives $\hat{D}_j$, in the present context they are required to verify the following properties:

(i) the $\hat{D}_j$ are linear operators;
(ii) they satisfy the Leibnitz rule: $\hat{D}_j(AB) = (\hat{D}_j A)B + A(\hat{D}_j B)$;
(iii) the integral of a derivative vanishes: $\mathrm{Tr}[\hat{D}_j A(\hat{x})] = 0$;
(iv) when $\theta \to$ const, $\hat{D}_j \to \partial_j$; this condition is not part of the formal definition of the derivatives, but we impose it in order to define actions that are comparable with their constant-θ counterparts.

Conditions (i)–(iii) are automatically satisfied if one uses *inner* derivations, i.e., those that can be written as commutators: $\hat{D}_j A \equiv [d_j, A]$, with $d_j \in \mathcal{A}$. When $\theta(\hat{x})$ does have an inverse (denoted by $\theta^{-1}(\hat{x})$), in $\mathcal{A}$, a suitable choice for the d_j is given by the expression:

$$d_j \equiv \frac{\mathrm{i}}{2}\epsilon_{jk}\{\theta^{-1}(\hat{x}), \hat{x}_k\}, \tag{3}$$

where $\{,\}$ denotes the anticommutator. Conditions (i)–(iii) are then valid (as for any inner derivation); regarding condition (iv), by acting on the generators of the algebra we see that

$$\hat{D}_j \hat{x}_k = \delta_{jk} + \frac{\mathrm{i}}{2}\epsilon_{jl}\{\hat{x}_l, [\theta^{-1}(\hat{x}), \hat{x}_k]\}. \tag{4}$$

Hence, condition (iv) is also fulfilled.

The final ingredient is the notion of adjoint conjugation. $A^\dagger$ is defined, as usual, by

$$\langle f|A^\dagger|g\rangle = \overline{\langle g|A|f\rangle} \qquad \forall\, f, g \in \mathcal{H}. \tag{5}$$

Since $d_j^\dagger = -d_j$, we see that the derivative of a Hermitian element of $\mathcal{A}$ will also be Hermitian.

We are now equipped to construct a noncommutative field theory in $(2+1)$ dimensions, the simplest example being that of a scalar field action S for a Hermitian field ϕ:

$$S = \int \mathrm{d}t\, \mathrm{Tr}\left[\frac{1}{2}D_\mu\phi(\hat{x}, t)D_\mu\phi(\hat{x}, t) + V(\phi)\right] \tag{6}$$

where $D_\mu \equiv (\partial_t, \hat{D}_1, \hat{D}_2)$ (the time coordinate is assumed to be commutative) and $V(\phi)$ is positive definite.

Although this is, indeed, a perfectly valid representation for a scalar field action on a noncommutative two-dimensional region, its form is inconvenient if one has in mind its use in concrete (e.g., perturbative) calculations. Besides, the quantization of the theory becomes problematic, and it is also rather difficult to compare results with those of its commutative counterpart.

To address this problem, in the following section we consider the equivalent description of the noncommutative theory in terms of functions equipped with a noncommutative $\star$-product.

3. Functional approach

In this setting, the dynamical fields are not operators, but rather elements of $C^\infty(\mathbb{R}^{2+1})$, and the (noncommutative) product between the operators in $\mathcal{A}$ is mapped onto a noncommutative $\star$-product.

This is, indeed, the idea behind the *deformation quantization* [20]; the tools and ideas needed to deal with this problem in a rather general setting have already been constructed (see, e.g., [20] and [14]). Explicit expressions within this approach, however, are difficult to derive, except when θ verifies certain regularity conditions, which allow the resulting $\star$-product to be expressed by an expansion in powers of θ and its derivatives. By 'regularity conditions' we mean that θ has to be sufficiently smooth for that expansion to converge. The measure of the smoothness is given by the relation between those derivatives and the (only) other dimensionful object, namely θ. More explicitly, we should have: $\partial_j\theta/\theta^{1/2} \ll 1$. We want to consider here situations where this condition for θ is not met (e.g., when θ vanishes with a nonzero derivative) so that an expansion in powers of θ and its derivatives is either not possible or unreliable. We do that by using a particular approach for the representation of the noncommutative algebra of operators over the space of functions, which bypasses the discussion on deformation quantization. In this way, we shall obtain a noncommutative $\star$-product which is valid even when such an expansion makes no sense. This $\star$-product is based on the operatorial approach of section 2, and it follows from the introduction of a specific mapping between operators and functions.

3.1. Normal-ordering and kernel representations

A one-to-one correspondence between operators $A(\hat{x})$ and functions $A(x)$ can be obtained by introducing a specific operator-ordering prescription. Here, we shall restrict the class of operators considered to those that can be put into a 'normal-ordered' form. We define that form by the condition that all the $\hat{x}_1$ have to appear to the left of all the $\hat{x}_2$ in the expansion of $A(\hat{x})$ in powers of $\hat{x}_1$ and $\hat{x}_2$. Namely, we shall consider the subspace $\mathcal{A}'$ of $\mathcal{A}$ that consists of all the operators $A(\hat{x})$ that can be represented as follows:

$$A(\hat{x}) = \sum_{m,n} a_{mn}\hat{x}_1^m\hat{x}_2^n \tag{7}$$

where the a_{mn} are (complex) constants[2].

For some particular forms of θ, any monomial in $\hat{x}_1$ and $\hat{x}_2$ can be converted into a normal-ordered form by performing a finite or infinite number of transpositions. For example, when θ is a normal-ordered formal series (what we shall assume), we can map any monomial in $\hat{x}_1, \hat{x}_2$ into normal order, albeit the monomial will be now equivalent to an infinite normal-ordered series. We shall later on consider a specific example which corresponds to a much simpler situation: a θ which depends only on the variable $\hat{x}_1$. In that case, every monomial is equivalent to a normal-ordered polynomial.

Thus for every operator $A(\hat{x})$ in $\mathcal{A}'$, we have $A(\hat{x}) = A_N(\hat{x})$ where $A_N(\hat{x})$ is its normal-ordered form. We assign to each $A(\hat{x})$ the (c-number) function $A_N(x)$, obtained by replacing in $A_N(\hat{x})$ the operators $\hat{x}_i$ by commutative coordinates x_i. We then have a one-to-one mapping $\mathcal{S}$:

$$A(\hat{x}) \to \mathcal{S}[A(\hat{x})] = A_N(x). \tag{8}$$

[2] $\mathcal{A}'$ and $\mathcal{A}$ can coincide or be isomorphic. That would be the case, for example, if the Poincaré–Birkhoff–Witt property were satisfied for $\mathcal{A}$.

To each operator $A(\hat{x})$ we can also associate another function: its 'mixed' integral kernel $A_K(x_1, x_2)$

$$A_K(x_1, x_2) = \langle x_1 | A(\hat{x}_1, \hat{x}_2) | x_2 \rangle, \tag{9}$$

where $\hat{x}_i |x_i\rangle = x_i |x_i\rangle$.

The relation between A_N and A_K is quite simple; indeed:

$$A_K(x_1, x_2) = \langle x_1 | x_2 \rangle A_N(x_1, x_2). \tag{10}$$

Since operator products are easily reformulated in terms of A_K, and we know its relation to A_N, we use this relation to define the $\star$-product.

3.2. *Definition of the $\star$-product*

To represent the algebra $\mathcal{A}$ on $C^\infty(\mathbb{R}^{2+1})$, we define the $\star$-product induced by the map (8):

$$A_N \star B_N \equiv \mathcal{S}[\mathcal{S}^{-1}(A_N)\mathcal{S}^{-1}(B_N)]. \tag{11}$$

On the other hand, equation (10) may be used to obtain an integral representation of the $\star$-product. Since

$$(AB)_K(x_1, x_2) = \int \mathrm{d}\tilde{x}_1 \, \mathrm{d}\tilde{x}_2 \, A_K(x_1, \tilde{x}_2) \langle \tilde{x}_2 | \tilde{x}_1 \rangle B_K(\tilde{x}_1, x_2), \tag{12}$$

we take (10) and (11) into account, to arrive at the expression

$$(A_N \star B_N)(x_1, x_2) = \int \mathrm{d}\tilde{x}_1 \, \mathrm{d}\tilde{x}_2 \frac{\langle x_1 | \tilde{x}_2 \rangle \langle \tilde{x}_2 | \tilde{x}_1 \rangle \langle \tilde{x}_1 | x_2 \rangle}{\langle x_1 | x_2 \rangle} A_N(x_1, \tilde{x}_2) B_N(\tilde{x}_1, x_2). \tag{13}$$

This product is evidently noncommutative, and it is also associative:

$$(A_N \star B_N) \star C_N = A_N \star (B_N \star C_N), \tag{14}$$

a property which can explicitly be verified by using the definition of the $\star$-product, or also by noting that the associativity of the operator product is inherited by the $\star$-product (by an application of the normal-order mapping).

Furthermore, it is immediate to prove that $(C^\infty(\mathbb{R}^{2+1}), \star)$, henceforth noted as $C^\infty_\star$, reproduces the structure of $\mathcal{A}$. Indeed, a straightforward calculation shows that

$$[x_1, x_2]_\star \equiv x_1 \star x_2 - x_2 \star x_1 = \mathrm{i}\theta_N(x). \tag{15}$$

Equation (13) is an integral representation of the noncommutative algebra (1), which depends on the function

$$F(x_1, x_2; \tilde{x}_1, \tilde{x}_2) = \frac{\langle x_1 | \tilde{x}_2 \rangle \langle \tilde{x}_2 | \tilde{x}_1 \rangle \langle \tilde{x}_1 | x_2 \rangle}{\langle x_1 | x_2 \rangle}. \tag{16}$$

Constructing $F(x; \tilde{x})$ for a general $\theta_N(x)$ is a very complicated problem; from equation (15) we may derive the integral equation

$$\int \mathrm{d}\tilde{x}_1 \, \mathrm{d}\tilde{x}_2 \, F(x_1, x_2; \tilde{x}_1, \tilde{x}_2) = x_1 x_2 - \mathrm{i}\theta_N(x_1, x_2). \tag{17}$$

Even an expansion in powers of $\theta(x)$ is quite involved if θ has a general dependence on (x_1, x_2). However, in section 4 we will show how to obtain explicit expressions for the simpler case $\theta = \theta(x_1)$.

3.3. Integral, derivatives and adjoints in $C^\infty_\star$

On the basis of the results of section 2, we construct the integral and derivatives now on $C^\infty_\star$. In what follows, to simplify the notation, we suppress the 'N' subscript when denoting a normal symbol.

If $\theta(x)$ is everywhere different from zero, two possible inner derivations are obtained by rewriting those of the operatorial formulation, namely,

$$D_j A(x) \equiv [d_j(x), A(x)]_\star, \tag{18}$$

where

$$d_j \equiv \frac{\mathrm{i}}{2}\epsilon_{jk}\{\theta^{-1}(x), x_k\}_\star. \tag{19}$$

In this equation θ^{-1} is not the usual inverse function, but rather the inverse w.r.t. the $\star$-product, i.e.,

$$\theta^{-1}(x) \equiv \mathcal{S}(\theta^{-1}(\hat{x})).$$

Since the D_j are $\star$-commutators, they are linear and obviously satisfy the Leibnitz rule,

$$D_j(A \star B) = D_j A \star B + A \star D_j B, \tag{20}$$

which is the $C^\infty_\star$ version of condition (ii) in section 2. However, using the explicit expression (13) for the $\star$-products in the derivatives, we conclude that

$$\int \mathrm{d}x_1\, \mathrm{d}x_2\; D_j(A(x)) \neq 0, \tag{21}$$

and

$$\int \mathrm{d}x_1\, \mathrm{d}x_2 (A \star B)(x) \neq \int \mathrm{d}x_1\, \mathrm{d}x_2 (B \star A)(x), \tag{22}$$

where $\int \mathrm{d}x_1\, \mathrm{d}x_2$ is the usual integration over $\mathbb{R}^2$ with a (flat) Euclidean metric. Hence, neither integration by parts (with respect to D_i) nor cyclicity would be valid if this definition of integral were used. Both of these properties, which are essential in the construction of a sensible field theory, can fortunately be satisfied if the factor $|\langle x_1|x_2\rangle|^2$ is included in the integration measure. Thus, we define the integral as

$$\int \mathrm{d}\mu(x) A(x) \equiv \int \mathrm{d}x_1\, \mathrm{d}x_2 |\langle x_1|x_2\rangle|^2 A(x). \tag{23}$$

The previous definition could also be derived from the operatorial trace, since one notes that

$$\mathrm{Tr} A(\hat{x}) = \int \mathrm{d}x_1\, \mathrm{d}x_2 |\langle x_1|x_2\rangle|^2 A_N(x). \tag{24}$$

Besides, the equalities

$$\int \mathrm{d}\mu(x) D_i(A(x)) = 0, \tag{25}$$

$$\int \mathrm{d}\mu(x)(A \star B)(x) = \int \mathrm{d}\mu(x)(B \star A)(x) \tag{26}$$

are simple consequences of equation (24).

On the other hand, the adjoint defined in section 2 can be represented in $C^\infty_\star$ by defining

$$A^\dagger(x) = \mathcal{S}[A^\dagger(\hat{x})]. \tag{27}$$

From (5), (9) and (10), (27) can be represented explicitly as

$$A^\dagger(x_1, x_2) = \int \mathrm{d}\tilde{x}_1\, \mathrm{d}\tilde{x}_2 \frac{\langle x_1|\tilde{x}_2\rangle\langle\tilde{x}_2|\tilde{x}_1\rangle\langle\tilde{x}_1|x_2\rangle}{\langle x_1|x_2\rangle} \bar{A}(\tilde{x}_1, \tilde{x}_2). \tag{28}$$

While in the Weyl ordering prescription the hermiticity of operators is tantamount to reality of functions, here $A^\dagger(\hat{x}) = A(\hat{x})$ translates into the condition:

$$A(x_1, x_2) = \int \mathrm{d}\tilde{x}_1\, \mathrm{d}\tilde{x}_2 \frac{\langle x_1|\tilde{x}_2\rangle\langle\tilde{x}_2|\tilde{x}_1\rangle\langle\tilde{x}_1|x_2\rangle}{\langle x_1|x_2\rangle} \bar{A}(\tilde{x}_1, \tilde{x}_2). \tag{29}$$

4. The case $\theta(x_1, x_2) = \theta(x_1)$

It is evident that knowledge of $\langle x_1|x_2\rangle$ plays a fundamental role in the definition of the $\star$-product previously introduced. In order to carry out a more detailed analysis, we restrict ourselves here to a particular case, tailored such that $\langle x_1|x_2\rangle$ can be evaluated exactly. A simple way to accomplish this is to consider the following form for $\theta(x)$:

$$[x_j, x_k]_\star = \mathrm{i}\epsilon_{jk}\theta(c_1x_1 + c_2x_2). \tag{30}$$

Under the redefinitions: $c_1x_1 + c_2x_2 \to x_1, x_2 \to x_2$, this relation can be equivalently written as

$$[x_j, x_k]_\star = \mathrm{i}\epsilon_{jk}\theta(x_1). \tag{31}$$

To obtain $\langle x_1|x_2\rangle$, we come back to the operatorial description of $\hat{x}_1$ and $\hat{x}_2$. The spectra of those operators can be found by representing them on the space of eigenfunctions of $\hat{x}_1$:

$$\hat{x}_1|x_1\rangle = x_1|x_1\rangle. \tag{32}$$

On that space, the Hermitian operator $\hat{x}_2$ is represented by

$$\hat{x}_2 = -\mathrm{i}\big(\theta(x_1)\partial_1 + \tfrac{1}{2}\theta'(x_1)\big), \tag{33}$$

where $\partial_1 \equiv \partial/\partial x_1, \theta'(x_1) \equiv \partial\theta/\partial x_1$. To find $\langle x_1|x_2\rangle$, we need the eigenvectors of $\hat{x}_2$ on the basis $\{|x_1\rangle\}$. Assuming that the operator $\theta(\hat{x}_1)$ is invertible, which is equivalent to saying that the function $\theta(x_1)$ has no zeros[3], we can solve the corresponding differential equation to find

$$\langle x_1|x_2\rangle = \frac{1}{\sqrt{2\pi}}\theta^{-1/2}(x_1)\, \exp\left[\mathrm{i}x_2 \int^{x_1} \mathrm{d}y_1\, \theta^{-1}(y_1)\right], \tag{34}$$

which has continuous normalization: $\langle x_2|x_2'\rangle = \delta(x_2 - x_2')$. The spectra of both operators $\hat{x}_1$ and $\hat{x}_2$ are the set of all the real numbers. This property is modified, as we shall see, when the condition on the zeros of θ is relaxed.

4.1. Properties of the $\star$-product

Since $|\langle x_1|x_2\rangle|^2 = \frac{1}{2\pi}\theta^{-1}(x_1)$, the integration measure in (23) becomes

$$\mathrm{d}\mu(x) \equiv \frac{1}{2\pi}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\theta^{-1}(x_1). \tag{35}$$

The noncommutative product of (13) reduces to

$$(A \star B)(x) = \int \mathrm{d}\mu(\tilde{x})\, \exp[\mathrm{i}(x_2 - \tilde{x}_2)\Delta g(x_1, \tilde{x}_1)]A(x_1, \tilde{x}_2)B(\tilde{x}_1, x_2), \tag{36}$$

where

$$g(x_1) \equiv \int^{x_1} \mathrm{d}y_1\, \theta^{-1}(y_1), \qquad \Delta g(x_1, \tilde{x}_1) \equiv g(\tilde{x}_1) - g(x_1). \tag{37}$$

[3] This condition will be relaxed later on.

By using some elementary algebra, we derive the useful relations:

$$A(x_1) \star B(x_1, x_2) = A(x_1)B(x_1, x_2) \tag{38}$$

$$A(x_1, x_2) \star B(x_2) = A(x_1, x_2)B(x_2) \tag{39}$$

$$A(x_2) \star B(x_1) = A(-\mathrm{i}\theta(x_1)\partial_1 + x_2)B(x_1). \tag{40}$$

Furthermore, (38)–(40) may be used to obtain an alternative expression for the $\star$-product. Writing the generic normal-ordered function as

$$A(x_1, x_2) = \sum_n \alpha_n^A(x_1)\beta_n^A(x_2), \tag{41}$$

we see that

$$A(x_1, x_2) \star B(x_1, x_2) = \sum_{n,m} \alpha_n^A(x_1)\beta_n^A(-\mathrm{i}\theta(x_1)\partial_1 + x_2)\alpha_m^B(x_1)\beta_m^B(x_2). \tag{42}$$

Either form (36) or (42) may prove to be more useful, depending on the context. For instance, if an expansion in powers of $\theta(x_1)$ and its derivatives is valid, equation (42) gives

$$\begin{aligned} A_N(x) \star B_N(x) &= A_N(x)B_N(x) - \mathrm{i}\theta(x_1)\partial_2 A_N(x)\partial_1 B_N(x) \\ &\quad - \tfrac{1}{2}\theta^2(x_1)\partial_2^2 A_N(x)\partial_1^2 B_N(x) - \tfrac{1}{2}\theta^2(x_1)\theta'(x_1)\partial_2^2 A_N(x)\partial_1 B_N(x) + \cdots. \end{aligned} \tag{43}$$

Let us conclude by considering the derivatives for the present case. From the operatorial construction, we have the general expression:

$$D_j A(x) = [d_j(x), A(x)]_\star, \tag{44}$$

where

$$d_j(x) = \frac{\mathrm{i}}{2}\epsilon_{jk}\{\theta^{-1}(x), x_k\}_\star. \tag{45}$$

Applying relations (38)–(40), we may simplify the expressions for the d_j for the particular case $\theta(x) = \theta(x_1)$:

$$\begin{aligned} d_1(x) &= \mathrm{i}\theta^{-1}(x_1)x_2 - \tfrac{1}{2}\theta^{-1}(x_1)\partial_1\theta(x_1) \\ d_2(x) &= -\mathrm{i}\theta^{-1}(x_1)x_1. \end{aligned} \tag{46}$$

Then the action of the D_1 derivative on $A(x_1, x_2)$ may be written as follows:

$$D_1 A(x) = \partial_1 A(x) + \mathrm{i}x_2[\theta^{-1}(x_1), A(x)]_\star - \tfrac{1}{2}[\theta^{-1}(x_1)\partial_1\theta(x_1), A(x)]_\star. \tag{47}$$

It is worth noting that, since θ depends only on x_1, $\theta^{-1}(x_1)$ coincides with the usual inverse function

On the other hand, D_2 is given by

$$D_2 A(x) = \mathrm{i}[\theta^{-1}(x_1)x_1, A(x)]_\star, \tag{48}$$

and it does not lead immediately to a similarly simple expression, involving a detached term with ∂_2. Indeed, after a little algebra one sees that

$$D_2 A(x) = (1 - x_1\theta^{-1}(x_1)\partial_1\theta(x_1))\partial_2 A(x) + \cdots \tag{49}$$

where the omitted terms involve higher powers of ∂_2 acting on A.

However, the problem of coping with the previous expression for D_2 can be entirely avoided by recalling that, since the noncommutative function $\theta(x)$ depends only on x_1, a simpler definition of a derivative should exist as a reflection of the invariance of θ under x_2 translations. Indeed, the outer derivative

$$D_2 A(x) \equiv \partial_2 A(x), \tag{50}$$

satisfies all the properties of a derivation:

$$\partial_2(A \star B) = \partial_2 A \star B + A \star \partial_2 B, \qquad \int d\mu(x)\partial_2 A(x) = 0 \tag{51}$$

for any $A(x)$ vanishing at infinity. We shall henceforth assume that D_2 stands for (50), while D_1 corresponds to (47).

Of course, D_1 and D_2 cannot be simultaneously diagonalized (a property that also holds true when both are *inner* derivatives). Their commutator reads

$$[D_1, D_2]A(x) = -\mathrm{i}[\theta^{-1}(x_1), A(x)]_\star, \tag{52}$$

which is akin to a noncommutative curvature. The relation between Poisson structure and curvature has been considered in [21].

This completes our discussion on the tools required to construct a field theory over $C_\star^\infty$. Again, the simplest case corresponds to the noncommutative generalization of a real scalar field action $S(\phi)$:

$$S(\phi) = l^2 \int dt\, d\mu(x) \left(\frac{1}{2} D_\nu\phi(x) \star D_\nu\phi(x) + V_\star(\phi)\right), \tag{53}$$

which is the functional transcription of the operatorial action (6). We have introduced the parameter l, with the dimensions of a length, in order to have a dimensionless action S. l can be naturally associated with the typical length defined by $\sqrt{\theta(x_1)}$. ϕ satisfies the constraint (29) and $V_\star[\phi(x)] = \mathcal{S}(V[\phi(\hat{x})])$ is assumed to be positive.

4.2. Interpretation

Formulae (47), (50) and (52) have an intuitive physical interpretation in terms of the Landau problem [9]. For a charged particle of mass m moving in the plane in the presence of a perpendicular magnetic field depending only on one of the coordinates: $B = B(x_1)$, the Lagrangian is

$$L = \tfrac{1}{2} m\dot{x}_i^2 - \dot{x}_i A_i(x).$$

Since $B = \partial_1 A_2 - \partial_2 A_1$, we can choose, for the vector potential,

$$A_1 = -x_2 B(x_1) + \varphi(x_1), \qquad A_2 = 0,$$

where $\varphi(x_1)$ accounts for the remanent gauge freedom. The mechanical momentum operators, defined as

$$\hat{\pi}_j = -\mathrm{i}\partial_j + A_j(\hat{x}),$$

have the commutation relations

$$[\hat{\pi}_1, \hat{\pi}_2] = -\mathrm{i}B(\hat{x}_1).$$

Their utility comes from the fact that the Hamiltonian is

$$H = \frac{1}{2m}\pi_i^2.$$

Remembering that the noncommutativity (31) is associated with the reduction to the lowest Landau level [22, 12] , we are naturally led to identify the D_j with the mechanical momenta: $D_j \to \mathrm{i}\hat{\pi}_j$. Then $-[\theta^{-1}(x_1),]_\star$ is interpreted as the noncommutative generalization of a magnetic field and the D_j correspond to the gauge choice

$$A_1 = x_2[\theta^{-1}(x_1),]_\star + \frac{\mathrm{i}}{2}[\theta^{-1}(x_1)\partial_1\theta(x_1),]_\star, \qquad A_2 = 0.$$

5. Boundary contribution

In this section we study the consequences of extending the previous formulae to a case in which $\theta(x_1)$ has a zero. As we will see, this has interesting physical consequences, which we shall study for the case of a theory defined by a noncommutative scalar field action.

Assuming that $\theta(x_1)$ has only one zero, at $x_1 = 0$, D_1 and $\int d\mu(x)$ are ill-defined at $x_1 = 0$; furthermore, from the eigenvalue equation for $\langle x_1|x_2\rangle$, one finds that $\langle x_1|x_2\rangle$ exists and it is unique only for $x_1 \in (0,\infty)$ or $x_1 \in (-\infty, 0)$, but not for the whole real axis. A restriction of the operators to only one of those intervals naturally suggests itself. We shall represent the operators $\hat{x}_1$ and $\hat{x}_2$ over the subspace corresponding to the eigenvalues in the interval $(0,\infty)$. Note that the presence of a zero in θ has led naturally to the existence of a boundary: $(x_1 = 0, x_2)$ in the configuration space. This boundary corresponds to the region where the coordinates commute, and it defines a (lower dimensional) commutative theory.

To deal with the singularities at $x_1 = 0$, we introduce a parameter ε such that $x_1 \in (\varepsilon,\infty)$, and $\varepsilon \to 0$ amounts to approaching the boundary (which cannot be exactly reached, since some operations would be ill-defined there). At the operatorial level, this restriction can be achieved by the introduction of a projection operator $P_\varepsilon(\hat{x}_1)$, such that $P_\varepsilon(x_1) = H(x_1-\varepsilon)$, where $H(x_1-\varepsilon)$ is Heaviside's step function. For instance, the trace over $x_1 \in (\varepsilon,\infty)$ can be 'regulated' (to avoid the boundary) as follows:

$$\mathrm{Tr}(P_\varepsilon(\hat{x}_1)A(\hat{x})). \tag{54}$$

Translating this into the functional language, this means to consider regulated integrals I_ε, of the form

$$I_\varepsilon = \int d\mu(x)P_\varepsilon(x_1)\star A(x), \tag{55}$$

with $d\mu(x)$ given by (35). Recalling (38), this integral can always be written as

$$I_\varepsilon = \int d\mu(x)P_\varepsilon(x_1)A(x). \tag{56}$$

On the other hand, the derivatives D_i defined in equations (47) and (50) are well defined on $x_1 \in (\varepsilon,\infty)$. However, due to the presence of the projector in the integration measure, the *regulated* integral of a derivative is no longer zero:

$$\int d\mu(x)P_\varepsilon(x_1)D_iA(x) = -\int d\,\mu(x)(D_iP_\varepsilon(x_1))A(x). \tag{57}$$

Since, from (47),

$$D_iP_\varepsilon(x_1) = \delta_{1i}\partial_1P_\varepsilon(x_1) = \delta_{1i}\delta(x_1-\varepsilon),$$

we arrive to

$$\int d\mu(x)P_\varepsilon(x_1)D_iA(x) = -\delta_{1i}\int \frac{dx_2}{2\pi\theta(\varepsilon)}A(x_1=\varepsilon,x_2). \tag{58}$$

Therefore, a boundary contribution is generated. This property is to be expected from the physical point of view, since there should be a positive 'jump' in the number of degrees of freedom when the theory becomes commutative.

Let us apply the previous procedure to a scalar field, whose regulated action is

$$S = \frac{l^2}{2}\int dt\,d\mu(x)P_\varepsilon(x_1)D_\nu\phi(x)\star D_\nu\phi(x), \tag{59}$$

where $D_\nu = (\partial_0, D_1, \partial_2)$. After integrating by parts, and applying (58), we see that

$$S = \frac{l^2}{2}\int dt\,d\mu(x)P_\varepsilon(x_1)\frac{1}{2}(\phi(x)\star D^2\phi(x) + D^2\phi(x)\star\phi(x)) + S_b \tag{60}$$

where

$$S_b \equiv -\frac{l^2}{4\pi\theta(\varepsilon)}\int \mathrm{d}t\,\mathrm{d}x_2 \frac{1}{2}\big(\phi(x)\star D_1\phi(x) + D_1\phi(x)\star\phi(x)\big)\Big|_{x_1=\varepsilon}. \tag{61}$$

The first ('bulk') term does not generate any boundary contribution. However, the second term S_b, which is a by-product of the zero at $x_1 = 0$, is a boundary term.

S_b is a (1 + 1)-dimensional action on the boundary $x_1 = \varepsilon$ which, in general, has a complicated dynamics. To derive a more explicit form, we take into account (42) and (47) to write

$$\begin{aligned} D_1\phi(x) &= \partial_1\phi(x) + \mathrm{i}x_2\theta^{-1}(x_1)\phi(x) - \mathrm{i}x_2\sum_r a_r(x_1)b_r(-\mathrm{i}\theta(x_1)\partial_1 + x_2)\theta^{-1}(x_1) \\ &\quad - \frac{1}{2}\partial_1 \ln\theta(x_1)\phi(x) + \frac{1}{2}\sum_r a_r(x_1)b_r(-\mathrm{i}\theta(x_1)\partial_1 + x_2)\partial_1\ln\theta(x_1), \end{aligned} \tag{62}$$

where we used the expansion

$$\phi(x_1, x_2) = \sum_r a_r(x_1)b_r(x_2). \tag{63}$$

Besides, we have

$$\phi(x)\star D_1\phi(x) = \int \mathrm{d}\mu(\tilde{x})P_\varepsilon(\tilde{x}_1)\exp[\mathrm{i}(x_2 - \tilde{x}_2)\Delta g(x_1, \tilde{x}_1)]\phi(x_1, \tilde{x}_2)D_1\phi(\tilde{x}_1, x_2), \tag{64}$$

and an analogous expression for $\phi(x)\star D_1\phi(x)$.

A non-trivial boundary contribution is then derived, whose explicit behaviour depends upon the precise form of $\theta(x_1)$. Note that the $\star$-product, when conveniently expanded, will introduce also derivatives with respect to x_2. This is illustrated by the following example.

5.1. Example

The simplest situation occurs when

$$\theta(x_1) = lgx_1, \tag{65}$$

where the dimensionless parameter g controls the 'strength' of the noncommutativity. Here we shall perform an expansion valid for $g \ll 1$, as this approximation serves to the purpose of exhibiting a local form for the term (61). For an arbitrary g, the term is of course non-local.

The first step is to compute

$$\Delta g(x_1, \tilde{x}_1) = (lg)^{-1}\ln\frac{\tilde{x}_1}{x_1}. \tag{66}$$

Substituting (65) in (62), we have

$$D_1\phi(x_1, x_2) = \partial_1\phi(x_1, x_2) + \frac{x_2 + \mathrm{i}lg/2}{x_1}\frac{\phi(x_1, x_2 + \mathrm{i}lg) - \phi(x_1, x_2)}{\mathrm{i}lg}. \tag{67}$$

When g is very small (at l fixed),

$$D_1\phi(x_1, x_2) = \partial_1\phi(x_1, x_2) + \frac{x_2}{x_1}\partial_2\phi(x_1, x_2) + \frac{\mathrm{i}lg}{2}\left[\frac{x_2}{x_1}\partial_2^2\phi(x_1, x_2) + \frac{1}{x_1}\partial_2\phi(x_1, x_2)\right] + \cdots. \tag{68}$$

In $\phi\star D_1\phi + D_1\phi\star\phi$ there is a factor

$$\exp[\mathrm{i}(x_2 - \tilde{x}_2)\Delta g(x_1, \tilde{x}_1)] = \exp[\mathrm{i}(lg)^{-1}(x_2 - \tilde{x}_2)\ln(\tilde{x}_1/x_1)].$$

so in the $g \to 0$ limit the stationary phase approximation is valid. To express $\phi \star D_1\phi + D_1\phi \star \phi$ as a local series in powers of g, we have to expand around x_2: $\tilde{x}_2 = x_2 + g\xi$. After some algebraic manipulations, we arrive to the leading contribution for $\varepsilon \to 0$:

$$S_b = \frac{1}{2}\frac{l}{4\pi g\varepsilon^2}\int \mathrm{d}t\,\mathrm{d}x_2 \left[\phi^2 + (gl)^2(\partial_2\phi)^2 - \frac{2}{3}(gl)^2 x_2\phi\partial_2^3\phi + O(g^2\varepsilon, g\varepsilon^2)\right], \tag{69}$$

where the field is evaluated at $x_1 = \varepsilon$. To get a reduced field with the proper ((1 + 1) dimensional) canonical dimension, we make the redefinition:

$$\tilde{\phi}(x_2) = \left(\frac{1}{4\pi}\right)^{1/2}\frac{l}{\varepsilon}\sqrt{gl}\phi(\epsilon, x_2). \tag{70}$$

Hence

$$S_b = \frac{1}{2}\int \mathrm{d}t\,\mathrm{d}x_2 \left[(\partial_2\tilde{\phi})^2 + \frac{1}{(gl)^2}\tilde{\phi}^2 - \frac{2}{3}x_2\tilde{\phi}\partial_2^3\tilde{\phi} + \cdots\right]. \tag{71}$$

Since the reality condition (29) for θ given by equation (65) can be written as

$$\phi(x) = \bar{\phi}(x) + O(g),$$

S_b contains a real (up to order g) mass term contribution in (1+1) dimensions, with a mass M inversely proportional to g, given by

$$M = \frac{1}{gl}. \tag{72}$$

The action is not translation invariant, something that can be understood as a relic of the existence of an external (non-constant) magnetic field. Furthermore, it is time independent, and it may be interpreted as proportional to the static energy of the boundary.

Note that this action comes from the first two terms in a small-g expansion for $\phi \star D_1\phi + D_1\phi \star \phi$, and it already has some physical information. The leading contribution, for example, goes like g^{-2}, and is precisely the kind of term that one would introduce to enforce the Dirichlet boundary condition $\phi(x_1 = 0, x_2) = 0$. The higher order corrections (also at order $1/\varepsilon$) yield terms containing derivatives of the scalar field on the boundary.

6. Conclusions

In this paper we have discussed some aspects of space-dependent planar noncommutativity, based on a particular mapping between normal-ordered operators and functions. The use of such a mapping has proved to be quite useful, since it allows one to derive explicit forms for the basic tools of the corresponding noncommutative field theory.

Guided by the operatorial description of section 2, in section 3 we defined an integral, derivatives, and adjointness on $C^\infty_\star$. A further restriction to the case $\theta(x_1, x_2) = \theta(x_1)$ allowed us to compute explicitly the $\star$-product obtaining the rather simple expression (36) and other useful relations (38)–(42). The results for that case are consistent with those of [15] (equations (20)–(28) of that paper), in the sense that the square root of the metric is related to θ in the same way we have found to be the case in the integration measure.

Equation (36) is valid for quite general functions $\theta(x_1)$. In particular, in section 5, we showed how to generalize our approach to a function $\theta(x)$ vanishing along the line $x_1 = 0$. This is a very interesting situation, since in the line $x_1 = 0$ there is a transition from a noncommutative theory to a commutative one. Of course, the transition is not continuous and many objects from the noncommutative theory are ill-defined over that region. The same situation appears when one takes the limit $\theta \to 0$ in the constant-θ case. Therefore, the zero creates a boundary in the configuration space that cannot be reached. We proved, starting from a noncommutative scalar field theory, that there are boundary contributions to the action, deriving the explicit form of its first few terms for the case $\theta(x_1) = \alpha x_1$.

A natural question that arises at this point refers to the relation between this approach to construct a noncommutative $\star$-product and deformation quantization results, as stated in [14]. If an expansion in powers of $\theta(x_1)$ and its derivatives is valid, then the $\star$-product defines a deformation quantization. Indeed, equation (43) defines, according to [14, 20], a star product.

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References

[1] Douglas M and Nekrasov N 2001 *Rev. Mod. Phys.* **73** 977–1029
[2] Szabo R 2003 *Phys. Rep.* **378** 207
[3] Castellani L 2000 *Class. Quantum Grav.* **17** 3377
[4] Susskind L 2001 *Preprint* hep-th/0101029
[5] Polychronakos A P 2001 *J. High Energy Phys.* JHEP04(2001)011
[6] Karabali D and Sakita B 2001 *Phys. Rev.* B **64** 245316
[7] Pasquier V 2000 *Phys. Lett.* B **490** 258
[8] Lee B H, Moon K and Rim C 2001 *Phys. Rev.* D **64** 085014
[9] Landau L and Lifshitz E 1977 *Quantum Mechanics* vol 3 3rd edn (Oxford: Butterworth-Heinemann)
[10] Girvin S M and Jach T 1984 *Phys. Rev.* B **29** 5617
[11] Ciccolini M, Fosco C D and Lopez A 2002 *J. Phys. A: Math. Gen.* **35** 10077
[12] Dunne G V, Jackiw R and Trugenberger C A 1990 *Phys. Rev* D **41** 661
[13] Kogan I I 1994 *Int. J. Mod. Phys.* A **9** 3887
[14] Kontsevich M 1997 Deformation quantization of Poisson manifolds: I *Preprint* q-alg/9709040
[15] Correa D H, Moreno E F and Schaposnik F A 2002 *Phys. Lett.* B **543** 235
[16] Cattaneo A and Felder G 2000 *Commun. Math. Phys.* **212** 591
[17] Cornalba L and Schiappa R 2002 *Commun. Math. Phys.* **225** 33
[18] Das A and Frenkel J 2004 *Phys. Rev.* D **69** 065017
[19] Madore J, Schraml S, Schupp P and Wess J 2000 *Eur. Phys. J.* C **16** 161
[20] Cattaneo A 2004 Formality and star products *Preprint* math.qa/0403135
[21] Madore J 1999 *Rep. Math. Phys.* **43** 231
[22] Fosco C D and Lopez A 2004 *J. Phys.: Condens. Matter* A **37** 4123